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Exercise 2

Value Functions, Dynamic Programming and Optimal Policies

1 Policy and Value Iteration

In this exercise you will implement important dynamic programming approaches for solving MDPs. You will use them to solve a simple gridworld MDP similar to the one shown in Figure 1. For quick reference, the respective algorithms are outlined in more detail in Figures 2, 3 and 4. [1]

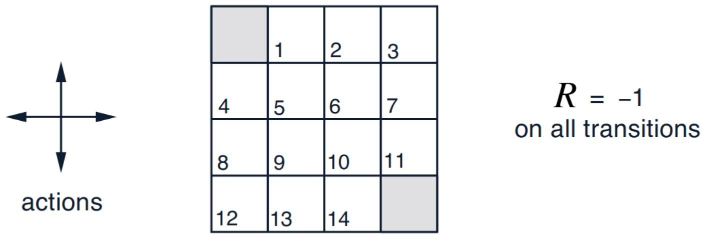

Figure 1: A gridworld MDP. Upper left and lower right states are terminal states.

Programming Tasks:

1. **Policy evaluation (one step)** Fill in code inside `policy_evaluation_one_step`, which performs exactly one step of policy evaluation (the inner part of the loop inside Figure 2).
2. **Policy evaluation** Using `policy_evaluation_one_step`, implement the method `policy_evaluation` which iteratively performs one step of policy evaluation until the changes in the value function estimate are smaller then some predefined threshold.
3. **Policy improvement** Continue with the `policy_improvement` method, which returns the optimal greedy policy w.r.t a given value function.
4. Utilizing your results from 1. - 3. implement:
   (a) the **policy iteration** (PI) algorithm (Figure 3) inside `policy_iteration`
   (b) the **value iteration** (VI) algorithm (Figure 4) inside `value_iteration` (*Hint:* You can either do this the *easy* way and reuse methods of 1. - 3. or do it the *efficient* way by closely following the algorithm outline of Figure 4).

[1] We recommend reading the first chapters of Suton ("Reinforcement Learning: An Introduction" by Richard S. Sutton and Andrew G. Barto, 1998) for an excellent introduction into the topic.

Input π, the policy to be evaluated
Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
 $\Delta \leftarrow 0$
 For each $s \in \mathcal{S}$:
  $v \leftarrow V(s)$
  $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
  $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$

Figure 2: Policy evaluation.

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
     $v \leftarrow V(s)$
     $V(s) \leftarrow \sum_{s',r} p(s',r|s,\pi(s))\big[r + \gamma V(s')\big]$
     $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
    $a \leftarrow \pi(s)$
    $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
    If $a \neq \pi(s)$, then *policy-stable* $\leftarrow$ $false$
   If *policy-stable*, then stop and return V and π; else go to 2

Figure 3: Policy iteration.

2 Optimal Policies

Try to answer the following questions concerning value functions and optimal policies.

Questions:

1. Imagine you know the optimal state-value function V of an MDP but you have no information about the state transition probabilities P. Could you derive the optimal policy from the state-value function V?

2. What would change, if you would have access to the optimal action-value function Q instead? [2]

3. Can there be multiple optimal deterministic policies for one MDP? If so, explain the conditions under which this would be the case.

[2] This is the fundamental idea of a very important method (or rather class of methods) that goes under the name of Q-Learning. The modern version of this utilized neural networks as an action-value function approximator, thus the name Deep Q-Learning (DQN) ("Playing Atari with Deep Reinforcement Learning" by Mnih et al., 2013). Stay tuned!

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
  $\Delta \leftarrow 0$
  For each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, π, such that
  $\pi(s) = \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$

Figure 4: Value iteration.